Kostenko E., Kuznichenko V. M., Lapshyn V. I. Generalized continuous linear model of international trade

The probability-based approach to the linear model of international trade based on the theory of Markov processes with continuous time is analysed. A generalized continuous model of international trade is built, in which the transition of the system from state to state is described by linear differential equations. The methodology of how to obtain the intensity matrices, which are differential in nature, is shown, and the same is done for their corresponding transition matrices for processes of purchasing and selling. In the process of the creation of the continuous model, functions and operations of matrices were used in addition to the Laplace transform, which gave the analytical form of the transition matrices, and therefore the expressions for the state vectors of the system. The obtained expressions simplify analysis and calculations in comparison to other methods. The values of the continuous transition matrices include in themselves the results of discrete model of international trade at moments in time proportional to the time step. The continuous model improves the quality of planning and the effectiveness of control of international trade agreements.

Key words: Markov chains, linear differential equations, z-transform, transform Laplace

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The probability-based approach to the linear model of international trade based on the theory of Markov processes with continuous time is analysed. A generalized continuous model of international trade is built, in which the transition of the system from state to state is described by linear differential equations. The methodology of how to obtain the intensity matrices, which are differential in nature, is shown, and the same is done for their corresponding transition matrices for processes of purchasing and selling. In the process of the creation of the continuous model, functions and operations of matrices were used in addition to the Laplace transform, which gave the analytical form of the transition matrices, and therefore the expressions for the state vectors of the system. The obtained expressions simplify analysis and calculations in comparison to other methods. The values of the continuous transition matrices include in themselves the results of discrete model of international trade at moments in time proportional to the time step. The continuous model improves the quality of planning and the effectiveness of control of international trade agreements.

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Formulation of the problem. One of the key elements of a country’s international activity is its trade, which also has a significant impact on its external politics. Integration processes in the global economy are related to the creation of transnational corporations, euroregions and regional unifications [1, 2], which require good planning and constant control to facilitate interaction between partners. This, in turn, increases the organizational standards of international trade. For these reasons, the improvement of models of international trade is very important.

Many Ukrainian scientists such as О. Г. Белорус, А. С. Гальчинский, Д. Г. Лукъяненко, Ю. В. Магагон and А. С. Филипенко [3–10] have studied Ukraine’s international trade – it’s characteristic values, it’s peculiarities and methods of its development. The studies over the last few years have shown the presence of a new step in the development of international trade relations in the post-crisis period. Data from 2010-2011 shows that the volume of world trade is on the rise. At the same time, the formation of an import structure from 2010-2011 shows that the volume of world trade is on the rise. At the same time, the formation of an import structure from 2010-2011 shows that the volume of world trade is on the rise.

It is important to note the comparative lack of articles on economic-mathematical models of international trade. The application of international logistical systems to better control material fluxes is analysed in [19]. The discrete linear model of international trade, based on the Markov Chain Method, was presented and analysed in [20–22] (deficit-less model) and in [23] (generalized model, which includes both deficit-inclusive and deficit-less models).

Formulation of the problem. The goal of the current article is the creation of a generalized continuous linear model for international trade from a probability-based approach. This model will allow for the simultaneous analysis of operations, specifically the buying and selling of goods, conducted between the members of a multi-sided trade agreement. This will provide the means to plan and control the continuous commodity-money relationships between trade partners.

Results of analysis. In the Markov Chain theory, the transition of a system from one state to another is, for discrete and continuous processes, described by equations (1) and (2) respectively:
Let’s show how to get the intensity $A_1$ and $A_2$, as well as their corresponding transition matrices $H_1$ and $H_2$ for continuous buying and selling processes in the simple case of three trade partners.

$$L_1 = B_1^* B_2^T = \begin{pmatrix}
1 & 1 & 1 \\
2 & 4 & 4 \\
\end{pmatrix}, \quad L_2 = B_2^* B_1 = \begin{pmatrix}
25 & 5 & 13 \\
48 & 24 & 48 \\
\end{pmatrix}$$

(7)

$$B_1 = \begin{pmatrix}
5 & 1 & 1 \\
6 & 12 & 12 \\
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
1 & 15 & 1 \\
32 & 16 & 32 \\
\end{pmatrix}$$

(8)

We will look at a discrete problem – the linear model of international trade, for which the stochastic ergodic transition matrices of a sales ($L_1$) and purchases ($L_2$) are defined in the following way (note that the processes of sales and purchases are considered to be unified) [23]:

$$\begin{pmatrix}
\frac{1}{2} & 0 & 1 \\
\frac{1}{4} & 0 & 1 \\
\end{pmatrix}^n + \begin{pmatrix}
\frac{1}{4} & 0 & 1 \\
\frac{1}{4} & 0 & 1 \\
\end{pmatrix}^n$$

(9)

$$\begin{pmatrix}
\frac{7}{24} & 5 & 1 \\
\frac{7}{24} & 5 & 1 \\
\end{pmatrix}^n + \begin{pmatrix}
\frac{7}{24} & 5 & 1 \\
\frac{7}{24} & 5 & 1 \\
\end{pmatrix}^n$$

(10)

Let’s build the continuous model of this problem. To do this, let’s first find the matrix $A_1$. The characteristic polynomial of the matrix $L_1$ will be written as $\Lambda_{L_1}$, and it is equal to the following:

$$\Delta_{L_1}(\lambda) = -\frac{1}{2} \lambda - \frac{1}{2} \lambda - \frac{1}{4} \lambda - \frac{1}{4}$$

(11)

All of the roots of the characteristic polynomial are simple. Because of this, the characteristic polynomial coincides with the minimal polynomial $\psi(\lambda) = \Delta_{L_1}(\lambda)$. The spectre of the matrix $L_1$ will be written as $\Lambda_{L_1}$, and it is equal to the following:

$$\Lambda_{L_1} = \left\{ \frac{1}{4}, -\frac{1}{2}, 1 \right\}$$

(12)

The function $f(\lambda)=\ln(\lambda)$ is defined on the spectrum of the matrix $L_1$. If the function $f(\lambda)$ is defined on the spectrum of the matrix $L_1$, then by definition

$$f(L_1) = g(\Lambda_{L_1})$$

(13)

This polynomial can be obtained in several ways. In our case, the lowest-order polynomial $g(\lambda)$ defined on the spectrum of the matrix $L_1$, will have the following form:

$$g(\lambda) = a\lambda^2 + b\lambda + c.$$ 

Let’s compose the system of linear algebraic equations:

$$\begin{pmatrix}
g(1) = f(1) = 0 = a + b + c \\
g(\frac{1}{2}) = f(\frac{1}{2}) = \ln(\frac{1}{2}) = -a + b + c \\
g(\frac{1}{4}) = f(\frac{1}{4}) = \ln(\frac{1}{4}) = -a + 2b + c \\
\end{pmatrix}$$

(14)

The solution of (14) has the following form: $a = 10/3\ln(1/2)$, $b = -6\ln(1/2)$, $c = 10/3\ln(1/2)$. Knowing $a,b,c$, we find
\[ A_t = \ln(L_t) = \left( \frac{8}{3}L_t^3 - 6L_t + \frac{10}{3} \right) \ln(\frac{1}{2}) = \ln(2) \begin{pmatrix} -\frac{5}{4} & 1 & \frac{3}{4} \\ 1 & -\frac{1}{2} & \frac{1}{4} \\ \frac{3}{4} & 1 & -\frac{5}{4} \end{pmatrix} \]

\[ sI - A_t = \begin{pmatrix} \frac{s - \frac{5}{2} \ln 2}{2} & -\frac{\ln 2}{2} & -\frac{3 \ln 2}{4} \\ -\frac{\ln 2}{4} & \frac{s + \frac{1}{2} \ln 2}{2} & -\frac{\ln 2}{4} \\ -\frac{3 \ln 2}{4} & -\frac{\ln 2}{2} & \frac{s + \frac{5}{2} \ln 2}{2} \end{pmatrix} \]

(15)

Let’s apply the Laplace transform to equation (5), thereby obtaining the matrix \( sI - A_t \):

\[ \begin{pmatrix} \frac{s^2 + \frac{7}{4} \ln 2 + \frac{1}{2} \ln 2^2}{s(s + \ln 2)(s + 2 \ln 2)} & \frac{1}{2} s \ln 2 + (\ln 2)^2 & \frac{3}{4} s \ln 2 + \frac{1}{2} \ln 2^2 \\ \frac{1}{4} s \ln 2 + \frac{1}{2} \ln 2^2 & \frac{s^2 + \frac{5}{2} s \ln 2 + (\ln 2)^2}{s(s + \ln 2)(s + 2 \ln 2)} & \frac{3}{4} s \ln 2 + \frac{1}{2} \ln 2^2 \\ \frac{3}{4} s \ln 2 + \frac{1}{2} \ln 2^2 & \frac{1}{2} s \ln 2 + (\ln 2)^2 & \frac{s^2 + \frac{7}{4} s \ln 2 + \frac{1}{2} \ln 2^2}{s(s + \ln 2)(s + 2 \ln 2)} \end{pmatrix} \]

(16)

The inverse matrix \((sI - A_t)^{-1}\) has the following form:

\[ (sI - A_t)^{-1} = \begin{pmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{pmatrix} + \frac{1}{s + \ln 2} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \]

(17)

Simplifying it, we obtain

\[ (sI - A_t)^{-1} = \frac{1}{s} \begin{pmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{pmatrix} + \frac{1}{s + \ln 2} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \]

(18)

Let’s assume that the matrix \( H_t(t) \) is the inverse transformation of the matrix \((sI - A_t)^{-1}\). Then, the inverse transformation transforms the equation (4) into

\[ \bar{p}_t(t) = \bar{p}_t(0)H_t(t) \]

(19)

Using the Laplace transform, we obtain

\[ \bar{p}_t(t) = \bar{p}_t(0) \left( \frac{1}{4} + e^{-\ln 2} \right) \begin{pmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \end{pmatrix} + e^{-2\ln 2} \begin{pmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \end{pmatrix} \]

(20)

Comparing (5) with (20), we see that \( H_t(t) \) defines the form of the matrix \( e^{At} \):

\[ H_t(t) = \begin{pmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \end{pmatrix} + e^{-\ln 2} \begin{pmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \end{pmatrix} + e^{-2\ln 2} \begin{pmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \end{pmatrix} \]

(21)

When \( t = n \), we obtain the following:

\[ H_n(n) = \begin{pmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \end{pmatrix} + e^{-\ln 2} \begin{pmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \end{pmatrix} + e^{-2\ln 2} \begin{pmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \end{pmatrix} = \]

\[ \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \]
Therefore, the expression (9) coincides with (20) when \( t = n \). This points to the fact that the sales transition matrix \( H_1(t) \) has been found. It is a component of the continuous model of international trade, and it includes in itself all of the values of the discrete model.

We will now obtain the second component of the continuous model – the matrix of intensity of transitions \( A_2 \) and the purchasing transition matrix \( H_2(t) \).

First of all, let’s find the characteristic polynomial of the matrix \( A_2 \):

\[
\Delta_{A_2}(\lambda) = \lambda - \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} -1 & -3 & -1 \\ -1 & -4 & -1 \\ -1 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = (\lambda - 1)(\lambda - 2) \begin{bmatrix} -1 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{bmatrix} \quad (23)
\]

All of the roots of the characteristic polynomial are simple, and thus the characteristic polynomial coincides with the minimal polynomial \( \psi(\lambda) = \Delta_{A_2}(\lambda) \). The spectrum of the matrix \( A_2 \) will be written as \( \Lambda_{A_2} \), and it is equal to:

\[
\Lambda_{A_2} = \left\{ \frac{1}{4}; \frac{1}{2}; 1 \right\} \quad (24)
\]

The function \( f(\lambda) = \ln(\lambda) \) is defined on the spectrum of the matrix \( A_2 \). As with the matrix \( L_1 \), we have, by definition,

\[
f(L_2) = g(L_2)
\quad (25)
\]

In our case, the lowest-order polynomial \( g(\lambda) \) defined on the spectrum of the matrix \( L_2 \) will have the following form:

\[
g(\lambda) = a\lambda^2 + b\lambda + c
\]

The system of linear algebraic equations for \( a, b, c \) coincides with (14), and their values are \( a = 8/3 \ln(1/2) \), \( b = -6\ln(1/2) \), \( c = 10/3 \ln(1/2) \).

In the analysed case, the matrix \( A_2 \) has the following form:

\[
\begin{pmatrix}
29 & 5 \\
7 & 7 \\
19 & 5
\end{pmatrix}
\quad (26)
\]

Let’s apply the Laplace transform to equation (5) and obtain the matrices \( sI - A_2 \) and \( (sI - A_2)^{-1} \):

\[
(sI - A_2)^{-1} = \begin{pmatrix}
\frac{s}{12} + \frac{\ln 2}{12} & -\frac{\ln 2}{24} & -\frac{\ln 2}{24} \\
-\frac{\ln 2}{24} & \frac{s}{12} + \frac{\ln 2}{12} & -\frac{s + 2\ln 2}{24} \\
-\frac{s + 2\ln 2}{24} & \frac{s + 2\ln 2}{24} & \frac{s + 2\ln 2}{24}
\end{pmatrix}
\quad (27)
\]
Simplifying it, we obtain
\[
(sI - A_2)^{-1} = \frac{1}{s} \begin{pmatrix}
  7 & 5 & 7 \\
  24 & 12 & 24 \\
  7 & 5 & 7 \\
  24 & 12 & 24 
\end{pmatrix} + \frac{1}{(s + \ln 2)} \begin{pmatrix}
  5 & 5 & 5 \\
  24 & 12 & 24 \\
  -7 & 7 & -7 \\
  24 & 12 & 24 
\end{pmatrix} + \frac{1}{(s + 2\ln 2)} \begin{pmatrix}
  1 & 0 & -1 \\
  2 & 0 & 1 \\
  0 & 0 & 0 \\
  1 & 0 & 1 
\end{pmatrix} \tag{29}
\]

Let’s assume that the matrix \( H_2(t) \) is the inverse transformation of the matrix \((sI - A_2)^{-1}\). Then, the inverse transformation transforms the equation (4) into
\[
\tilde{p}_2(t) = \tilde{p}_2(0)H_2(t) \tag{30}
\]

Using the Laplace transform, we obtain
\[
\tilde{p}_2(t) = \tilde{p}_2(0) \begin{pmatrix}
  7 & 5 & 7 \\
  24 & 12 & 24 \\
  7 & 5 & 7 \\
  24 & 12 & 24 
\end{pmatrix} + e^{-t\ln 2} \begin{pmatrix}
  5 & 5 & 5 \\
  24 & 12 & 24 \\
  -7 & 7 & -7 \\
  24 & 12 & 24 
\end{pmatrix} + e^{-2t\ln 2} \begin{pmatrix}
  1 & 0 & -1 \\
  2 & 0 & 1 \\
  0 & 0 & 0 \\
  1 & 0 & 1 
\end{pmatrix} \tag{31}
\]

Comparing (5) with (31), we see that \( H_2(t) \) defines the form of the matrix \( e^{At} \):
\[
H_2(t) = \begin{pmatrix}
  7 & 5 & 7 \\
  24 & 12 & 24 \\
  7 & 5 & 7 \\
  24 & 12 & 24 
\end{pmatrix} + e^{-t\ln 2} \begin{pmatrix}
  5 & 5 & 5 \\
  24 & 12 & 24 \\
  -7 & 7 & -7 \\
  24 & 12 & 24 
\end{pmatrix} + e^{-2t\ln 2} \begin{pmatrix}
  1 & 0 & -1 \\
  2 & 0 & 1 \\
  0 & 0 & 0 \\
  1 & 0 & 1 
\end{pmatrix} \tag{32}
\]

When \( t = n \), we obtain the following from (32):
\[
H_2(n) = \begin{pmatrix}
  7 & 5 & 7 \\
  24 & 12 & 24 \\
  7 & 5 & 7 \\
  24 & 12 & 24 
\end{pmatrix} + e^{-n\ln 2} \begin{pmatrix}
  5 & 5 & 5 \\
  24 & 12 & 24 \\
  -7 & 7 & -7 \\
  24 & 12 & 24 
\end{pmatrix} + e^{-2n\ln 2} \begin{pmatrix}
  1 & 0 & -1 \\
  2 & 0 & 1 \\
  0 & 0 & 0 \\
  1 & 0 & 1 
\end{pmatrix} \tag{33}
\]

Therefore, the expression (9) coincides with (20) when \( t = n \). This points to the fact that the sales transition matrix \( H_1(t) \) has been found. It is a component of the continuous model of international trade, and it includes in itself all of the values of the discrete model.

From (33) it follows that the formulas (10) and (32) coincide at \( t = n \). This points to the fact that the purchase transition matrix has been found. It is a component of the continuous model of international trade, and it includes in itself all of the values of the discrete model.

Conclusions. The generalized continuous linear model of linear trade has been built. It allows us to simultaneously analyse operations, specifically the buying and selling of goods, conducted between the members of a multi-sided trade agreement. It also allows us to obtain the balance in their trade relations. The method of derivation of the intensity matrices and their corresponding transition matrices for the buying and selling processes was shown. The values of the continuous transition matrices include all of the results of the discrete model of international trade at the moments of time proportional to the time step. The continuous model improves the quality of planning and the effectiveness of control in international trade relations.

**ЛІТЕРАТУРА**


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